

REVERSES OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY FOR n -TUPLES OF COMPLEX NUMBERS

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ABSTRACT. Some new reverses of the Cauchy-Bunyakovsky-Schwarz inequality for n -tuples of real and complex numbers related to Cassels and Shisha-Mond results are given.

1. INTRODUCTION

Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two positive n -tuples with the property that there exists the positive numbers m_i, M_i ($i = 1, 2$) such that

$$(1.1) \quad 0 < m_1 \leq a_i \leq M_1 < \infty \quad \text{and} \quad 0 < m_2 \leq b_i \leq M_2 < \infty,$$

for each $i \in \{1, \dots, n\}$.

The following reverses of the Cauchy-Bunyakovsky-Schwarz (CBS) inequality are well known in the literature:

(1) **Pólya-Szegő's inequality** [8]

$$(1.2) \quad \frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2;$$

(2) **Shisha-Mond's inequality** [9]

$$(1.3) \quad \frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leq \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}} \right)^2;$$

(3) **Ozeki's inequality** [7]

$$(1.4) \quad \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k \right)^2 \leq \frac{1}{4} n^2 (M_1 M_2 - m_1 m_2)^2;$$

(4) **Diaz-Metcalf's inequality** [1]

$$(1.5) \quad \sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^n a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^n a_k b_k.$$

If the weight $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive n -tuple, then we have the following inequalities, which are also well known.

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5. **Cassel's inequality** [10]

If the positive n -tuples $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition

$$(1.6) \quad 0 < m \leq \frac{a_k}{b_k} \leq M < \infty \quad \text{for each } k \in \{1, \dots, n\},$$

where m, M are given, then

$$(1.7) \quad \frac{\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \leq \frac{(M+m)^2}{4mM}.$$

6. **Grueb-Reinboldt's inequality** [4]

If $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ satisfy the condition (1.1), then

$$(1.8) \quad \frac{\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2}.$$

7. **Generalised Diaz-Metcalf inequality** [1] (see also [6, p. 123])

If $u, v \in [0, 1]$ and $v \leq u$, $u + v = 1$ and (1.6) holds, then one has the inequality

$$(1.9) \quad u \sum_{k=1}^n w_k b_k^2 + vmM \sum_{k=1}^n w_k a_k^2 \leq (vm + uM) \sum_{k=1}^n w_k a_k b_k.$$

8. **Klamkin-McLenaghan's inequality** [5]

If $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ satisfy (1.6), then we have the inequality

$$(1.10) \quad \sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2 - \left(\sum_{k=1}^n w_k a_k b_k \right)^2 \leq \left(\sqrt{M} - \sqrt{m} \right)^2 \sum_{k=1}^n w_k a_k b_k \sum_{k=1}^n w_k a_k^2.$$

For other reverse results of the (CBS)-inequality, see the recent survey online [3].

The main aim of this paper is to point out some new reverse inequalities of the classical Cauchy-Bunyakovsky-Schwarz result for both real and complex n -tuples.

2. SOME REVERSES OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY

The following result holds.

Theorem 1. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n) \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$. If $b_i \neq 0$, $i \in \{1, \dots, n\}$ and there exists the constant $\alpha \in \mathbb{K}$ and $r > 0$ such that for any $k \in \{1, \dots, n\}$

$$(2.1) \quad \frac{a_k}{b_k} \in \bar{D}(\alpha, r) := \{z \in \mathbb{K} \mid |z - \alpha| \leq r\},$$

then we have the inequality

$$(2.2) \quad \sum_{k=1}^n p_k |a_k|^2 + \left(|\alpha|^2 - r^2 \right) \sum_{k=1}^n p_k |b_k|^2 \leq 2 \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right].$$

The constant $c = 2$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. From (2.1) we have $|a_k - \alpha \bar{b}_k|^2 \leq r |b_k|^2$ for each $k \in \{1, \dots, n\}$, which is clearly equivalent to

$$(2.3) \quad |a_k|^2 + \left(|\alpha|^2 - r^2 \right) |b_k|^2 \leq 2 \operatorname{Re} [\bar{\alpha} (a_k b_k)]$$

for each $k \in \{1, \dots, n\}$.

Multiplying (2.3) with $p_k \geq 0$ and summing over k from 1 to n , we deduce the first inequality in (1.2). The second inequality is obvious.

To prove the sharpness of the constant 2, assume that under the hypothesis of the theorem there exists a constant $c > 0$ such that

$$(2.4) \quad \sum_{k=1}^n p_k |a_k|^2 + (|\alpha|^2 - r^2) \sum_{k=1}^n p_k |b_k|^2 \leq c \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right],$$

provided $\frac{a_k}{b_k} \in \bar{D}(\alpha, r)$, $k \in \{1, \dots, n\}$.

Assume that $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $b_1 = b_2 = 1$, $\alpha = r > 0$ and $a_2 = 2r$, $a_1 = 0$. Then $\left| \frac{a_2}{b_2} - \alpha \right| = r$, $\left| \frac{a_1}{b_1} - \alpha \right| = r$ showing that the condition (2.1) holds. For these choices, the inequality (2.4) becomes $2r^2 \leq cr^2$, giving $c \geq 2$. ■

The case where the disk $\bar{D}(\alpha, r)$ does not contain the origin, i.e., $|\alpha| > r$, provides the following interesting reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

Theorem 2. *Let $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{p}}$ as in Theorem 1 and assume that $|\alpha| > r > 0$. Then we have the inequality*

$$(2.5) \quad \sum_{k=1}^n p_k |a_k|^2 \sum_{k=1}^n p_k |b_k|^2 \leq \frac{1}{|\alpha|^2 - r^2} \left\{ \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] \right\}^2 \\ \leq \frac{|\alpha|^2}{|\alpha|^2 - r^2} \left| \sum_{k=1}^n p_k a_k b_k \right|^2.$$

The constant $c = 1$ in the first and second inequality is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Since $|\alpha| > r$, we may divide (2.2) by $\sqrt{|\alpha|^2 - r^2} > 0$ to obtain

$$(2.6) \quad \frac{1}{\sqrt{|\alpha|^2 - r^2}} \sum_{k=1}^n p_k |a_k|^2 + \sqrt{|\alpha|^2 - r^2} \sum_{k=1}^n p_k |b_k|^2 \\ \leq \frac{2}{\sqrt{|\alpha|^2 - r^2}} \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right].$$

On the other hand, by the use of the following elementary inequality

$$(2.7) \quad \frac{1}{\beta} p + \beta q \geq 2\sqrt{pq} \quad \text{for } \beta > 0 \text{ and } p, q \geq 0,$$

we may state that

$$(2.8) \quad 2 \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} \\ \leq \frac{1}{\sqrt{|\alpha|^2 - r^2}} \sum_{k=1}^n p_k |a_k|^2 + \sqrt{|\alpha|^2 - r^2} \sum_{k=1}^n p_k |b_k|^2.$$

Utilising (2.6) and (2.8), we deduce

$$\left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|\alpha|^2 - r^2}} \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right],$$

which is clearly equivalent to the first inequality in (2.6).

The second inequality is obvious.

To prove the sharpness of the constant, assume that (2.5) holds with a constant $c > 0$, i.e.,

$$(2.9) \quad \sum_{k=1}^n p_k |a_k|^2 \sum_{k=1}^n p_k |b_k|^2 \leq \frac{c}{|\alpha|^2 - r^2} \left\{ \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] \right\}^2$$

provided $\frac{a_k}{b_k} \in \bar{D}(\alpha, r)$ and $|\alpha| > 1$.

For $n = 2$, $b_2 = b_1 = 1$, $p_1 = p_2 = \frac{1}{2}$, $a_2, a_1 \in \mathbb{R}$, $\alpha, r > 0$ and $\alpha > r$, we get from (2.9) that

$$(2.10) \quad \frac{a_1^2 + a_2^2}{2} \leq \frac{c\alpha^2}{\alpha^2 - r^2} \left(\frac{a_1 + a_2}{2} \right)^2.$$

If we choose $a_2 = \alpha + r$, $a_1 = \alpha - r$, then $|a_i - \alpha| \leq r$, $i = 1, 2$ and by (2.10) we deduce

$$\alpha^2 + r^2 \leq \frac{c\alpha^4}{\alpha^2 - r^2},$$

which is clearly equivalent to

$$(c - 1)\alpha^4 + r^4 \geq 0 \quad \text{for } \alpha > r > 0.$$

If in this inequality we choose $\alpha = 1$, $r = \varepsilon \in (0, 1)$ and let $\varepsilon \rightarrow 0+$, then we deduce $c \geq 1$. ■

The following corollary is a natural consequence of the above theorem.

Corollary 1. *Under the assumptions of Theorem 2, we have the following additive reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$(2.11) \quad 0 \leq \sum_{k=1}^n p_k |a_k|^2 \sum_{k=1}^n p_k |b_k|^2 - \left| \sum_{k=1}^n p_k a_k b_k \right|^2 \\ \leq \frac{r^2}{|\alpha|^2 - r^2} \left| \sum_{k=1}^n p_k a_k b_k \right|^2.$$

The constant $c = 1$ is best possible in the sense mentioned above.

Remark 1. *If in Theorem 1, we assume that $|\alpha| = r$, then we obtain the inequality:*

$$(2.12) \quad \sum_{k=1}^n p_k |a_k|^2 \leq 2 \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] \\ \leq 2 |\alpha| \left| \sum_{k=1}^n p_k a_k b_k \right|.$$

The constant 2 is sharp in both inequalities.

We also remark that, if $r > |\alpha|$, then (2.2) may be written as

$$(2.13) \quad \begin{aligned} \sum_{k=1}^n p_k |a_k|^2 &\leq (r^2 - |\alpha|^2) \sum_{k=1}^n p_k |b_k|^2 + 2 \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] \\ &\leq (r^2 - |\alpha|^2) \sum_{k=1}^n p_k |b_k|^2 + 2 |\alpha| \left| \sum_{k=1}^n p_k a_k b_k \right|. \end{aligned}$$

The following reverse of the Cauchy-Bunyakovsky-Schwarz inequality also holds.

Theorem 3. Let $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{p}}$ be as in Theorem 1 and assume that $\alpha \in \mathbb{K}$, $\alpha \neq 0$ and $r > 0$. Then we have the inequalities

$$(2.14) \quad \begin{aligned} 0 &\leq \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^n p_k a_k b_k \right| \\ &\leq \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\alpha}}{|\alpha|} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] \\ &\leq \frac{1}{2} \cdot \frac{r^2}{|\alpha|} \sum_{k=1}^n p_k |b_k|^2. \end{aligned}$$

The constant $\frac{1}{2}$ is best possible in the sense mentioned above.

Proof. From Theorem 1, we have

$$(2.15) \quad \sum_{k=1}^n p_k |a_k|^2 + |\alpha|^2 \sum_{k=1}^n p_k |b_k|^2 \leq 2 \operatorname{Re} \left[\bar{\alpha} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] + r^2 \sum_{k=1}^n p_k |b_k|^2.$$

Since $\alpha \neq 0$, we can divide (2.15) by $|\alpha|$, getting

$$(2.16) \quad \begin{aligned} \frac{1}{|\alpha|} \sum_{k=1}^n p_k |a_k|^2 + \sum_{k=1}^n p_k |b_k|^2 \\ \leq 2 \operatorname{Re} \left[\frac{\bar{\alpha}}{|\alpha|} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] + \frac{r^2}{|\alpha|} \sum_{k=1}^n p_k |b_k|^2. \end{aligned}$$

Utilising the inequality (2.7), we may state that

$$(2.17) \quad 2 \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} \leq \frac{1}{|\alpha|} \sum_{k=1}^n p_k |a_k|^2 + |\alpha| \sum_{k=1}^n p_k |b_k|^2.$$

Making use of (2.16) and (2.17), we deduce the second inequality in (2.14).

The first inequality in (2.14) is obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that there exists a $c > 0$ such that

$$(2.18) \quad \begin{aligned} \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\alpha}}{|\alpha|} \left(\sum_{k=1}^n p_k a_k b_k \right) \right] \\ \leq c \cdot \frac{r^2}{|\alpha|} \sum_{k=1}^n p_k |b_k|^2, \end{aligned}$$

provided $\left| \frac{a_k}{b_k} - \alpha \right| \leq r$, $\alpha \neq 0$, $r > 0$.

If we choose $n = 2$, $\alpha > 0$, $b_1 = b_2 = 1$, $a_1 = \alpha + r$, $a_2 = \alpha - r$, then from (2.18) we deduce

$$(2.19) \quad \sqrt{r^2 + \alpha^2} - \alpha \leq c \frac{r^2}{\alpha}.$$

If we multiply (2.19) with $\sqrt{r^2 + \alpha^2} + \alpha > 0$ and then divide it by $r > 0$, we deduce

$$(2.20) \quad 1 \leq \frac{\sqrt{r^2 + \alpha^2} + \alpha}{\alpha} \cdot c$$

for any $r > 0$, $\alpha > 0$.

If in (2.20) we let $r \rightarrow 0+$, then we get $c \geq \frac{1}{2}$, and the sharpness of the constant is proved. ■

3. A CASSELS TYPE INEQUALITY FOR COMPLEX NUMBERS

The following result holds.

Theorem 4. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n) \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$. If $b_i \neq 0$, $i \in \{1, \dots, n\}$ and there exist the constants $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $\Gamma \neq \gamma$, so that either

$$(3.1) \quad \left| \frac{a_k}{b_k} - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for each } k \in \{1, \dots, n\},$$

or, equivalently,

$$(3.2) \quad \operatorname{Re} \left[\left(\Gamma - \frac{a_k}{b_k} \right) \left(\frac{\bar{a}_k}{b_k} - \bar{\gamma} \right) \right] \geq 0 \quad \text{for each } k \in \{1, \dots, n\}$$

holds, then we have the inequalities

$$(3.3) \quad \sum_{k=1}^n p_k |a_k|^2 \sum_{k=1}^n p_k |b_k|^2 \leq \frac{1}{2 \operatorname{Re}(\Gamma\bar{\gamma})} \left\{ \operatorname{Re} \left[(\bar{\gamma} + \bar{\Gamma}) \sum_{k=1}^n p_k a_k b_k \right] \right\}^2 \\ \leq \frac{|\Gamma + \gamma|^2}{4 \operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{k=1}^n p_k a_k b_k \right|^2.$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible in (3.3).

Proof. The fact that the relations (3.1) and (3.2) are equivalent follows by the simple fact that for $z, u, U \in \mathbb{C}$, the following inequalities are equivalent

$$\left| z - \frac{u + U}{2} \right| \leq \frac{1}{2} |U - u|$$

and

$$\operatorname{Re}[(u - z)(\bar{z} - \bar{u})] \geq 0.$$

Define $\alpha = \frac{\gamma + \Gamma}{2}$ and $r = \frac{1}{2} |\Gamma - \gamma|$. Then

$$|\alpha|^2 - r^2 = \frac{|\Gamma + \gamma|^2}{4} - \frac{|\Gamma - \gamma|^2}{4} = \operatorname{Re}(\Gamma\bar{\gamma}) > 0.$$

Consequently, we may apply Theorem 2, and the inequalities (3.3) are proved.

The sharpness of the constants may be proven in a similar way to that in the proof of Theorem 2, and we omit the details. ■

The following additive version also holds.

Corollary 2. *With the assumptions in Theorem 4, we have*

$$(3.4) \quad \sum_{k=1}^n p_k |a_k|^2 \sum_{k=1}^n p_k |b_k|^2 - \left| \sum_{k=1}^n p_k a_k b_k \right|^2 \leq \frac{|\Gamma - \gamma|^2}{4 \operatorname{Re}(\Gamma \bar{\gamma})} \left| \sum_{k=1}^n p_k a_k b_k \right|^2.$$

The constant $\frac{1}{4}$ is also best possible.

Remark 2. *With the above assumptions and if $\operatorname{Re}(\Gamma \bar{\gamma}) = 0$, then by the use of Remark 1, we may deduce the inequality*

$$(3.5) \quad \sum_{k=1}^n p_k |a_k|^2 \leq \operatorname{Re} \left[(\bar{\gamma} + \bar{\Gamma}) \sum_{k=1}^n p_k a_k b_k \right] \leq |\Gamma + \gamma| \left| \sum_{k=1}^n p_k a_k b_k \right|.$$

If $\operatorname{Re}(\Gamma \bar{\gamma}) < 0$, then, by Remark 1, we also have

$$(3.6) \quad \begin{aligned} \sum_{k=1}^n p_k |a_k|^2 &\leq -\operatorname{Re}(\Gamma \bar{\gamma}) \sum_{k=1}^n p_k |b_k|^2 + \operatorname{Re} \left[(\bar{\Gamma} + \bar{\gamma}) \sum_{k=1}^n p_k a_k b_k \right] \\ &\leq -\operatorname{Re}(\Gamma \bar{\gamma}) \sum_{k=1}^n p_k |b_k|^2 + |\Gamma + \gamma| \left| \sum_{k=1}^n p_k a_k b_k \right|. \end{aligned}$$

Remark 3. *If $a_k, b_k > 0$ and there exist the constants $m, M > 0$ ($M > m$) with*

$$(3.7) \quad m \leq \frac{a_k}{b_k} \leq M \quad \text{for each } k \in \{1, \dots, n\},$$

then, obviously (3.1) holds with $\gamma = m$, $\Gamma = M$, also $\Gamma \bar{\gamma} = Mm > 0$ and by (3.3) we deduce

$$(3.8) \quad \sum_{k=1}^n p_k a_k^2 \sum_{k=1}^n p_k b_k^2 \leq \frac{(M+m)^2}{4mM} \left(\sum_{k=1}^n p_k a_k b_k \right)^2,$$

that is, Cassels' inequality.

4. A SHISHA-MOND TYPE INEQUALITY FOR COMPLEX NUMBERS

The following result holds.

Theorem 5. *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n) \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$. If $b_i \neq 0$, $i \in \{1, \dots, n\}$ and there exist the constants $\gamma, \Gamma \in \mathbb{K}$ such that $\Gamma \neq \gamma, -\gamma$ and either*

$$(4.1) \quad \left| \frac{a_k}{\bar{b}_k} - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for each } k \in \{1, \dots, n\},$$

or, equivalently,

$$(4.2) \quad \operatorname{Re} \left[\left(\Gamma - \frac{a_k}{b_k} \right) \left(\frac{\bar{a}_k}{\bar{b}_k} - \bar{\gamma} \right) \right] \geq 0 \quad \text{for each } k \in \{1, \dots, n\},$$

holds, then we have the inequalities

$$\begin{aligned}
 (4.3) \quad 0 &\leq \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^n p_k a_k b_k \right| \\
 &\leq \left(\sum_{k=1}^n p_k |a_k|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k |b_k|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \sum_{k=1}^n p_k a_k b_k \right] \\
 &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{k=1}^n p_k |b_k|^2.
 \end{aligned}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Follows by Theorem 3 on choosing $\alpha = \frac{\gamma + \Gamma}{2} \neq 0$ and $r = \frac{1}{2} |\Gamma - \gamma| > 0$.

The proof for the best constant follows in a similar way to that in the proof of Theorem 3 and we omit the details. ■

Remark 4. If $a_k, b_k > 0$ and there exists the constants $m, M > 0$ ($M > m$) with

$$(4.4) \quad m \leq \frac{a_k}{b_k} \leq M \quad \text{for each } k \in \{1, \dots, n\},$$

then we have the inequality

$$\begin{aligned}
 (4.5) \quad 0 &\leq \left(\sum_{k=1}^n p_k a_k^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^n p_k b_k^2 \right)^{\frac{1}{2}} - \sum_{k=1}^n p_k a_k b_k \\
 &\leq \frac{1}{4} \cdot \frac{(M - m)^2}{(M + m)} \sum_{k=1}^n p_k b_k^2.
 \end{aligned}$$

The constant $\frac{1}{4}$ is best possible. For $p_k = \frac{1}{n}$, $k \in \{1, \dots, n\}$, we recapture the result from [2, Theorem 5.21] that has been obtained from a reverse inequality due to Shisha and Mond [8].

5. FURTHER REVERSES OF THE (CBS)-INEQUALITY

The following result holds.

Theorem 6. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n) \in \mathbb{K}^n$ and $r > 0$ such that for $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$

$$(5.1) \quad \sum_{i=1}^n p_i |b_i - \bar{a}_i|^2 \leq r^2 < \sum_{i=1}^n p_i |a_i|^2.$$

Then we have the inequality

$$\begin{aligned}
 (5.2) \quad 0 &\leq \sum_{i=1}^n p_i |a_i|^2 \sum_{i=1}^n p_i |b_i|^2 - \left| \sum_{i=1}^n p_i a_i b_i \right|^2 \\
 &\leq \sum_{i=1}^n p_i |a_i|^2 \sum_{i=1}^n p_i |b_i|^2 - \left[\operatorname{Re} \left(\sum_{i=1}^n p_i a_i b_i \right) \right]^2 \\
 &\leq r^2 \sum_{i=1}^n p_i |b_i|^2.
 \end{aligned}$$

The constant $c = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. From the first condition in (5.1), we have

$$\sum_{i=1}^n p_i \left[|b_i|^2 - 2 \operatorname{Re}(b_i a_i) + |a_i|^2 \right] \leq r^2,$$

giving

$$(5.3) \quad \sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2 - r^2 \leq 2 \operatorname{Re} \left(\sum_{i=1}^n p_i a_i b_i \right).$$

Since, by the second condition in (5.1) we have

$$\sum_{i=1}^n p_i |a_i|^2 - r^2 > 0,$$

we may divide (5.3) by $\sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2} > 0$, getting

$$(5.4) \quad \frac{\sum_{i=1}^n p_i |b_i|^2}{\sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2} \leq \frac{2 \operatorname{Re}(\sum_{i=1}^n p_i a_i b_i)}{\sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2}}.$$

Utilising the elementary inequality

$$(5.5) \quad \frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq} \quad \text{for } p, q \geq 0 \text{ and } \alpha > 0,$$

we may write that

$$(5.6) \quad 2\sqrt{\sum_{i=1}^n p_i |b_i|^2} \leq \frac{\sum_{i=1}^n p_i |b_i|^2}{\sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2}.$$

Combining (5.5) with (5.6) we deduce

$$(5.7) \quad \sqrt{\sum_{i=1}^n p_i |b_i|^2} \leq \frac{\operatorname{Re}(\sum_{i=1}^n p_i a_i b_i)}{\sqrt{\sum_{i=1}^n p_i |a_i|^2 - r^2}}.$$

Taking the square in (5.7), we obtain

$$\sum_{i=1}^n p_i |b_i|^2 \left(\sum_{i=1}^n p_i |a_i|^2 - r^2 \right) \leq \left[\operatorname{Re} \left(\sum_{i=1}^n p_i a_i b_i \right) \right]^2,$$

giving the third inequality in (5.2).

The other inequalities are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that there exists a constant $c > 0$ such that

$$(5.8) \quad \sum_{i=1}^n p_i |a_i|^2 \sum_{i=1}^n p_i |b_i|^2 - \left[\operatorname{Re} \left(\sum_{i=1}^n p_i a_i b_i \right) \right]^2 \leq cr^2 \sum_{i=1}^n p_i |b_i|^2,$$

provided

$$\sum_{i=1}^n p_i |b_i - \bar{a}_i|^2 \leq r^2 < \sum_{i=1}^n p_i |a_i|^2.$$

Let $r = \sqrt{\varepsilon}$, $\varepsilon \in (0, 1)$, $a_i, e_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i |a_i|^2 = \sum_{i=1}^n p_i |e_i|^2 = 1$ and $\sum_{i=1}^n p_i a_i e_i = 0$. Put $b_i = \overline{a_i} + \sqrt{\varepsilon} e_i$. Then, obviously

$$\sum_{i=1}^n p_i |b_i - \overline{a_i}|^2 = r^2, \quad \sum_{i=1}^n p_i |a_i|^2 = 1 > r$$

and

$$\begin{aligned} \sum_{i=1}^n p_i |b_i|^2 &= \sum_{i=1}^n p_i |a_i|^2 + \varepsilon \sum_{i=1}^n p_i |e_i|^2 = 1 + \varepsilon, \\ \operatorname{Re} \left(\sum_{i=1}^n p_i a_i b_i \right) &= \sum_{i=1}^n p_i |a_i|^2 = 1 \end{aligned}$$

and thus

$$\sum_{i=1}^n p_i |a_i|^2 \sum_{i=1}^n p_i |b_i|^2 - \left[\operatorname{Re} \left(\sum_{i=1}^n p_i a_i b_i \right) \right]^2 = \varepsilon.$$

Using (5.8), we may write

$$\varepsilon \leq c\varepsilon(1 + \varepsilon) \quad \text{for } \varepsilon \in (0, 1),$$

giving $1 \leq c(1 + \varepsilon)$ for $\varepsilon \in (0, 1)$. Making $\varepsilon \rightarrow 0+$, we deduce $c \geq 1$. ■

The following result also holds.

Theorem 7. Let $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in \mathbb{K}^n$, $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$ and $\gamma, \Gamma \in \mathbb{K}$ such that $\operatorname{Re}(\gamma\bar{\Gamma}) > 0$ and either

$$(5.9) \quad \sum_{i=1}^n p_i \operatorname{Re}[(\Gamma\bar{y}_i - x_i)(\bar{x}_i - \bar{\gamma}y_i)] \geq 0,$$

or, equivalently,

$$(5.10) \quad \sum_{i=1}^n p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot \bar{y}_i \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2.$$

Then we have the inequalities

$$\begin{aligned} (5.11) \quad \sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 &\leq \frac{1}{4} \cdot \frac{\{\operatorname{Re}[(\bar{\Gamma} + \bar{\gamma}) \sum_{i=1}^n p_i x_i y_i]\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{i=1}^n p_i x_i y_i \right|^2. \end{aligned}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

Proof. Define $b_i = x_i$ and $a_i = \frac{\bar{\Gamma} + \bar{\gamma}}{2} \cdot y_i$ and $r = \frac{1}{2} |\Gamma - \gamma| \left(\sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}}$. Then, by (5.10)

$$\begin{aligned} \sum_{i=1}^n p_i |b_i - \overline{a_i}|^2 &= \sum_{i=1}^n p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot \bar{y}_i \right|^2 \\ &\leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2 = r^2, \end{aligned}$$

showing that the first condition in (5.1) is fulfilled.

We also have

$$\begin{aligned} \sum_{i=1}^n p_i |a_i|^2 - r^2 &= \sum_{i=1}^n p_i \left| \frac{\Gamma + \gamma}{2} \right|^2 |y_i|^2 - \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2 \\ &= \operatorname{Re}(\Gamma \bar{\gamma}) \sum_{i=1}^n p_i |y_i|^2 > 0 \end{aligned}$$

since $\operatorname{Re}(\gamma \bar{\Gamma}) > 0$, and thus the condition in (5.1) is also satisfied.

Using the second inequality in (5.2), one may write

$$\begin{aligned} &\sum_{i=1}^n p_i \left| \frac{\Gamma + \gamma}{2} \right|^2 |y_i|^2 \sum_{i=1}^n p_i |x_i|^2 - \left[\operatorname{Re} \sum_{i=1}^n p_i \left(\frac{\bar{\Gamma} + \bar{\gamma}}{2} \right) y_i x_i \right]^2 \\ &\leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2 \sum_{i=1}^n p_i |x_i|^2, \end{aligned}$$

giving

$$\frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4} \sum_{i=1}^n p_i |y_i|^2 \sum_{i=1}^n p_i |x_i|^2 \leq \frac{1}{4} \operatorname{Re} \left[(\bar{\Gamma} + \bar{\gamma}) \sum_{i=1}^n p_i x_i y_i \right]^2,$$

which is clearly equivalent to the first inequality in (5.11).

The second inequality in (5.11) is obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that the first inequality in (5.11) holds with a constant $c > 0$, i.e.,

$$(5.12) \quad \sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \leq C \cdot \frac{\left\{ \operatorname{Re} [(\bar{\Gamma} + \bar{\gamma}) \sum_{i=1}^n p_i x_i y_i] \right\}^2}{\operatorname{Re}(\Gamma \bar{\gamma})},$$

provided $\operatorname{Re}(\gamma \bar{\Gamma}) > 0$ and either (5.9) or (5.10) holds.

Assume that $\Gamma, \gamma > 0$ and let $x_i = \gamma \bar{y}_i$. Then (5.9) holds true and by (5.12) we deduce

$$\gamma^2 \left(\sum_{i=1}^n p_i |y_i|^2 \right)^2 \leq C \frac{(\Gamma + \gamma)^2 \gamma^2 \left(\sum_{i=1}^n p_i |y_i|^2 \right)^2}{\Gamma \gamma},$$

giving

$$(5.13) \quad \Gamma \gamma \leq C (\Gamma + \gamma)^2 \quad \text{for any } \Gamma, \gamma > 0.$$

Let $\varepsilon \in (0, 1)$ and choose in (5.13) $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon > 0$ to get $1 - \varepsilon^2 \leq 4C$ for any $\varepsilon \in (0, 1)$. Letting $\varepsilon \rightarrow 0+$, we deduce $C \geq \frac{1}{4}$ and the sharpness of the constant is proved.

Finally, we note that the conditions (5.9) and (5.10) are equivalent since in an inner product space $(H, \langle \cdot, \cdot \rangle)$ for any vectors $x, z, Z \in H$ one has $\operatorname{Re} \langle Z - x, x - z \rangle \geq 0$ iff $\left\| x - \frac{z+Z}{2} \right\| \leq \frac{1}{2} \|Z - z\|$ [1]. We omit the details. ■

6. MORE REVERSES OF THE (CBS)-INEQUALITY

The following result holds.

Theorem 8. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n) \in \mathbb{K}^n$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$. If $r > 0$ and the following condition is satisfied

$$(6.1) \quad \sum_{i=1}^n p_i |b_i - \bar{a}_i|^2 \leq r^2,$$

then we have the inequalities

$$(6.2) \quad \begin{aligned} 0 &\leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n p_i a_i b_i \right| \\ &\leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n p_i \operatorname{Re}(a_i b_i) \right| \\ &\leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}(a_i b_i) \\ &\leq \frac{1}{2} r^2. \end{aligned}$$

The constant $\frac{1}{2}$ is best possible in (6.2) in the sense that it cannot be replaced by a smaller constant.

Proof. The condition (6.1) is clearly equivalent to

$$(6.3) \quad \sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2 \leq 2 \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) + r^2.$$

Using the elementary inequality

$$(6.4) \quad 2 \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2$$

and (6.3), we deduce

$$(6.5) \quad 2 \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} \leq 2 \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) + r^2,$$

giving the last inequality in (6.2). The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that

$$(6.6) \quad 0 \leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) \leq cr^2$$

for any $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \mathbb{K}^n$ and $r > 0$ satisfying (6.1).

Assume that $\bar{\mathbf{a}}, \bar{\mathbf{e}} \in H$, $\bar{\mathbf{e}} = (e_1, \dots, e_n)$ with $\sum_{i=1}^n p_i |a_i|^2 = \sum_{i=1}^n p_i |e_i|^2 = 1$ and $\sum_{i=1}^n p_i a_i e_i = 0$. If $r = \sqrt{\varepsilon}$, $\varepsilon > 0$, and if we define $\bar{\mathbf{b}} = \bar{\mathbf{a}} + \sqrt{\varepsilon} \bar{\mathbf{e}}$ where $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_n) \in \mathbb{K}^n$, then $\sum_{i=1}^n p_i |b_i - \bar{a}_i|^2 = \varepsilon = r^2$, showing that the condition (6.1) is fulfilled.

On the other hand,

$$\begin{aligned}
 & \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) \\
 &= \left(\sum_{i=1}^n p_i |\overline{a_i} + \sqrt{\varepsilon} e_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}[(\overline{a_i} + \sqrt{\varepsilon} e_i) a_i] \\
 &= \left(\sum_{i=1}^n p_i |a_i|^2 + \varepsilon \sum_{i=1}^n |e_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i |a_i|^2 \\
 &= \sqrt{1 + \varepsilon} - 1.
 \end{aligned}$$

Utilizing (6.6), we conclude that

$$(6.7) \quad \sqrt{1 + \varepsilon} - 1 \leq c\varepsilon \quad \text{for any } \varepsilon > 0.$$

Multiplying (6.7) by $\sqrt{1 + \varepsilon} + 1 > 0$ and thus dividing by $\varepsilon > 0$, we get

$$(6.8) \quad (\sqrt{1 + \varepsilon} - 1) c \geq 1 \quad \text{for any } \varepsilon > 0.$$

Letting $\varepsilon \rightarrow 0+$ in (6.8), we deduce $c \geq \frac{1}{2}$, and the theorem is proved. ■

Finally, the following result also holds.

Theorem 9. Let $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in \mathbb{K}^n$, $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$, and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq \gamma, -\gamma$, so that either

$$(6.9) \quad \sum_{i=1}^n p_i \operatorname{Re}[(\Gamma \bar{y}_i - x_i)(\bar{x}_i - \bar{\gamma} y_i)] \geq 0,$$

or, equivalently,

$$(6.10) \quad \sum_{i=1}^n p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot \bar{y}_i \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2$$

holds. Then we have the inequalities

$$\begin{aligned}
 (6.11) \quad 0 &\leq \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n p_i x_i y_i \right| \\
 &\leq \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n p_i \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i \right] \right| \\
 &\leq \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i \right] \\
 &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^n p_i |y_i|^2.
 \end{aligned}$$

The constant $\frac{1}{4}$ in the last inequality is best possible.

Proof. Consider $b_i = x_i$, $a_i = \frac{\bar{\Gamma} + \bar{\gamma}}{2} \cdot y_i$, $i \in \{1, \dots, n\}$ and

$$r := \frac{1}{2} (\Gamma - \gamma) \left(\sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}}.$$

Then, by (6.10), we have

$$\sum_{i=1}^n p_i |b_i - \bar{a}_i|^2 = \sum_{i=1}^n p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot y_i \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2 = r^2$$

showing that (6.1) is valid.

By the use of the last inequality in (6.2), we have

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i \left| \frac{\Gamma + \gamma}{2} \right|^2 |y_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{2} x_i y_i \right] \\ &\leq \frac{1}{8} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2. \end{aligned}$$

Dividing by $\frac{1}{2} |\Gamma + \gamma| > 0$, we deduce

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i \right] \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^n p_i |y_i|^2, \end{aligned}$$

which is the last inequality in (6.11).

The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that there exists a constant $c > 0$, such that

$$\begin{aligned} (6.12) \quad &\left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i \right] \\ &\leq c \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^n p_i |y_i|^2, \end{aligned}$$

provided either (6.9) or (6.10) holds.

Let $n = 2$, $\bar{\mathbf{y}} = (1, 1)$, $\bar{\mathbf{x}} = (x_1, x_2) \in \mathbb{R}^2$, $\bar{\mathbf{p}} = (\frac{1}{2}, \frac{1}{2})$ and $\Gamma, \gamma > 0$ with $\Gamma > \gamma$. Then by (6.12) we deduce

$$(6.13) \quad \sqrt{2} \sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leq 2c \frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}.$$

If $x_1 = \Gamma$, $x_2 = \gamma$, then $(\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0$, showing that the condition (6.9) is valid for $n = 2$ and $\bar{\mathbf{p}}$, $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$ as above. Replacing x_1 and x_2 in (6.13), we deduce

$$(6.14) \quad \sqrt{2} \sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \leq 2c \frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}.$$

If in (6.14) we choose $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon$ with $\varepsilon \in (0, 1)$, we deduce

$$(6.15) \quad \sqrt{1 + \varepsilon^2} - 1 \leq 2c\varepsilon^2.$$

Finally, multiplying (6.15) with $\sqrt{1 + \varepsilon^2} + 1 > 0$ and then dividing by ε^2 , we deduce

$$(6.16) \quad 1 \leq 2c \left(\sqrt{1 + \varepsilon^2} + 1 \right) \quad \text{for any } \varepsilon > 0.$$

Letting $\varepsilon \rightarrow 0+$ in (6.16), we get $c \geq \frac{1}{4}$, and the sharpness of the constant is proved. ■

Remark 5. *The integral version may be stated in a canonical way. The corresponding inequalities for integrals will be considered in another work devoted to positive linear functionals with complex values that is in preparation.*

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